COMPACT RIEMANN SURFACES 100 POINTS

Notes.
(a) There are a total of 105 points in this paper. You will be awarded at most 100.
(b) Justify all your steps. Use only those results that have been proved in class unless you have been asked to prove the same.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) For a Riemann surface X,
$\mathcal{O}_X$ = the sheaf of holomorphic functions on X,
$\mathcal{E}_X$ = the sheaf of complex-valued C^∞-functions on X,
Ω_X = the sheaf of holomorphic 1-forms on X.
(e) $\mathbb{P}^1$ = the Riemann sphere.

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1. [20 points] Let $\Gamma \subset \mathbb{C}$ be a lattice. Define $\mathbb{C}/\Gamma$ as a topological space and describe how to give a complex structure on it. Prove that $\mathbb{C}/\Gamma$ is a compact Riemann surface.

2. [15 points] Let $\Gamma \subset \mathbb{C}$ be a lattice. Find the universal covering space of $\mathbb{C}/\Gamma$. Using this or otherwise prove that any holomorphic map $\mathbb{P}^1 \to \mathbb{C}/\Gamma$ is constant.

3. [20 points] Define what it means for a covering map $f\colon Y \to X$ of Riemann surfaces to be Galois. Prove that if $p\colon \widetilde{X} \to X$ is a universal cover then it is Galois and $\mathrm{Deck}(\widetilde{X}/X)$ is isomorphic to the fundamental group $\pi_1(X)$.

4. [15 points] Let X be a Riemann surface. For an open set $U \subset X$, let $\mathcal{F}(U) := \mathcal{O}^*(U)/\exp \mathcal{O}(U)$. Verify that $\mathcal{F}$, with the usual restriction maps, is a presheaf which is not a sheaf in general, i.e., find an example of X and $\mathcal{F}$ for which $\mathcal{F}$ as defined above is not a sheaf.

5. [15 points] Find the universal cover of $\mathbb{C}^*$. If $p\colon Y \to \mathbb{C}^*$ is the universal cover and θ is the holomorphic 1-form dz/z on $\mathbb{C}^*$, find $p^*\theta$.

6. [20 points] Let $X = \mathbb{C}/\Gamma$ where $\Gamma \subset \mathbb{C}$ is a lattice. Prove that there is a holomorphic 1-form $\omega \in \Omega(X)$ such that ω generates $T_x^{1,0}X$ for every $x \in X$.